JOURNAL OF APPROXIMATION THEORY **70**, 94–114 (1992)

Bounds on Multivariate Polynomials and Exponential Error Estimates for Multiquadric Interpolation

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Communicated by Nira Dyn

Received June 22, 1990; revised March 26, 1991

A class of multivariate scattered data interpolation methods which includes the so-called multiquadrics is considered. Pointwise error bounds are given in terms of several parameters including a parameter d which, roughly speaking, measures the spacing of the points at which interpolation occurs. In the multiquadric case these estimates are $O(\lambda^{1/d})$ as $d \to 0$, where λ is a constant which satisfies $0 < \lambda < 1$. An essential ingredient in this development which may be of independent interest is a bound on the size of a polynomial over a cube in R^n in terms of its values on a discrete subset which is scattered in a sufficiently uniform manner.

1. INTRODUCTION

Let h be a continuous function on R^n which is conditionally positive definite of order m. Given data (x_j, f_j), $j = 1, ..., N$, where $X = \{x_1, ..., x_N\}$ is a subset of points in R^n and the f_j's are real or complex numbers, the so-called h *spline* interpolant of these data is the function s defined by

$$s(x) = p(x) + \sum_{j=1}^{N} c_j h(x - x_j), \tag{1}$$

* Both authors were partially supported by a grant from the Air Force Office of Scientific Research, AFOSR-86-0145.

0021-9045/92 $5.00

where $p(x)$ is a polynomial in $\mathscr{P}_{m-1}$ and the c_j's are chosen so that

$$\sum_{j=1}^{N} c_j q(x_j) = 0 \tag{2}$$

for all polynomials q in $\mathscr{P}_{m-1}$ and

$$p(x_i) + \sum_{j=1}^{N} c_j h(x_i - x_j) = f_i, \qquad i = 1, ..., N. \tag{3}$$

Here $\mathscr{P}_{m-1}$ denotes the class of those polynomials of R^n of degree $\leqslant m-1$.

It is well known that the system of equations (2) and (3) has a unique solution when X is a determining set for $\mathscr{P}_{m-1}$ and h is strictly conditionally positive definite. For more details see [7]. Thus, in this case, the interpolant $s(x)$ is well defined.

We remind the reader that X is said to be a determining set for $\mathscr{P}_{m-1}$ if p is in $\mathscr{P}_{m-1}$ and p vanishes on X implies that p is identically zero.

If h is the function defined by the formula

$$h(x) = -\sqrt{1+|x|^2}, \tag{4}$$

where $|x|$ is the Euclidean norm of x, then $m=1$ and the corresponding method of interpolation defined by (1), (2), (3), and (4) is often referred to as the multiquadric method. This and closely related methods are currently quite fashionable, see [4, 10].

In an earlier paper [8] we obtained bounds on the pointwise difference between a function f and the h spline which agrees with f on a subset X of R^n. These estimates involve a parameter d that measures the spacing of the points in X and are $O(d^l)$ as $d \to 0$ where l depends on h. The results of the present paper imply that for certain h's, which include (4), the estimates can be improved to $O(\lambda^{1/d})$ as $d \to 0$, where λ is a constant which satisfies $0 < \lambda < 1$. The conditions on f are the same as those in [8].

1.1. *A Bound for Multivariate Polynomials*

A key ingredient in the development of our estimates is the following lemma which gives a bound on the size of a polynomial on a cube in R^n in terms of its values on a discrete subset which is scattered in a sufficiently uniform manner. This result may be of independent interest.

LEMMA 1. *For $n = 1, 2, ...,$ define γ_n by the formulas $\gamma_1 = 2$ and, if $n > 1$, $\gamma_n = 2n(1+\gamma_{n-1})$. Let Q be a cube in R^n that is subdivided into q^n identical subcubes. Let Y be a set of q^n points obtained by selecting a point from each of those subcubes. If $q \geqslant \gamma_n(k+1)$, then for all p in $\mathscr{P}_k$*

$$\sup_{x \in Q} |p(x)| \leqslant e^{2n\gamma_n(k+1)} \sup_{y \in Y} |p(y)|.$$

We remark that it is not essential for the set Y to intersect every subcube of Q as hypothesized above. A variant of this lemma where Y intersects a certain percentage of these subcubes can be found in Subsection 3.3.

Note that it follows from Lemma 1 that Y is a determining set of $\mathscr{P}_k$. The estimate in the lemma is roughly equivalent to a bound on the Lebesque constant for Lagrange interpolation. In the cases where Y is regularly distributed in Q this bound can be derived by more traditional methods; see [1, 2].

1.2. *A Variational Framework for Interpolation*

The precise statement of our estimates concerning h splines requires a certain amount of technical notation and terminology which is identical to that used in [8]. For the convenience of the reader we recall several basic notions.

The space of complex valued functions on R^n that are compactly supported and infinitely differentiable is denoted by $\mathscr{D}$. The Fourier transform of a function ϕ in $\mathscr{D}$ is

$$\hat{\phi}(\xi) = \int e^{-i\langle x, \xi\rangle}\, \phi(x)\, dx.$$

A continuous function h is conditionally positive definite of order m if

$$\int h(x) \phi * \tilde{\phi}(x)\, dx \geqslant 0 \tag{5}$$

holds whenever $\phi = p(D)\psi$ with ψ in $\mathscr{D}$ and $p(D)$ a linear homogeneous constant coefficient differential operator of order m. Here $\tilde{\phi}(x) = \overline{\phi(-x)}$ and $*$ denotes the convolution product

$$\phi_1 * \phi_2(t) = \int \phi_1(x)\, \phi_2(t-x)\, dx.$$

Note that (5) can be rewritten as

$$\iint h(x-y)\, \phi(x)\, \overline{\phi(y)}\, dx\, dy \geqslant 0.$$

In what follows h will always denote a continuous conditionally positive definite function of order m. The Fourier transform of such distributions uniquely determines a positive Borel measure μ on $R^n \sim \{0\}$ and constants a_γ, $|\gamma| = 2m$ as follows: For all $\psi \in \mathscr{D}$

$$\int h(x)\,\psi(x)\,dx = \int \left\{\hat{\psi}(\xi) - \hat{\chi}(\xi) \sum_{|\gamma| < 2m} D^\gamma \hat{\psi}(0)\,\frac{\xi^\gamma}{\gamma!}\right\} d\mu(\xi)$$

$$+ \sum_{|\gamma| \leqslant 2m} D^\gamma \hat{\psi}(0)\,\frac{a_\gamma}{\gamma!}, \tag{6}$$

where for every choice of complex numbers c_α, $|\alpha| = m$,

$$\sum_{|\alpha| = m} \sum_{|\beta| = m} a_{\alpha+\beta} c_\alpha \bar{c}_\beta \geqslant 0. \tag{7}$$

Here χ is a function in $\mathscr{D}$ such that $1 - \hat{\chi}(\xi)$ has a zero of order $2m+1$ at $\xi = 0$; both of the integrals $\int_{0<|\xi|<1} |\xi|^{2m}\,d\mu(\xi)$, $\int_{|\xi| \geqslant 1} d\mu(\xi)$ are finite. The choice of χ affects the value of the coefficients a_γ for $|\gamma| < 2m$.

Our variational framework for interpolation is supplied by a space we denote by $\mathscr{C}_{h,m}$. If

$$\mathscr{D}_m = \left\{\phi \in \mathscr{D}\colon \int x^\alpha \phi(x)\,dx = 0 \text{ for all } |\alpha| < m\right\}$$

then $\mathscr{C}_{h,m}$ is the class of those continuous functions f which satisfy

$$\left|\int f(x)\,\phi(x)\,dx\right| \leqslant c(f) \left\{\int h(x-y)\,\phi(x)\,\overline{\phi(y)}\,dx\,dy\right\}^{1/2} \tag{8}$$

for some constant $c(f)$ and all ϕ in $\mathscr{D}_m$. If $f \in \mathscr{C}_{h,m}$ let $\|f\|_h$ denote the smallest constant $c(f)$ for which (8) is true. Recall that $\|f\|_h$ is a semi-norm and $\mathscr{C}_{h,m}$ is a semi-Hilbert space; in the case $m = 0$ it is a norm and a Hilbert space respectively. Elements f in $\mathscr{C}_{h,m}$ are of the form

$$f = f_1 + f_2,$$

where the Fourier transform of f_1 is given by

$$\hat{f}_1(\xi) = g(\xi)\,d\mu(\xi)$$

with g in $L^2(d\mu)$ and f_2 is a polynomial of degree m.

Given a function f in $\mathscr{C}_{h,m}$ and a subset X of R^n there is an element s of minimal $\mathscr{C}_{h,m}$ norm which is equal to f on X. If X is a determining set for $\mathscr{P}_{m-1}$ then s is unique. We refer to such s as the h spline interpolant of f on X. In the case when X is a finite subset of R^n as considered in beginning of this introduction the h spline s is given by (1), where $f(x_i) = f_i$, $i = 1, ..., N$, See [7] for more details.

1.3. *Exponential Error Estimates*

Our basic theorem concerns how well s approximates f in regions Ω where X provides sufficient coverage. In other words, we are interested in bounds on the quantity

$$\frac{|f(x)-s(x)|}{\|f\|_h}, \tag{9}$$

where x is in Ω; the estimates should be in terms of parameters which measure how closely X covers Ω. For example, the parameter $d=d(\Omega, X)$ defined by

$$d(\Omega, X)=\sup_{y\in\Omega}\ \inf_{x\in X}|y-x|$$

is one such measure.

In [8] we showed that in many cases the quantity in (9) is $O(d^k)$ as $d\to 0$, where k is a constant whose maximum value is determined by h. In this paper we restrict our attention to h's whose corresponding measures μ defined by (6) satisfy certain moment condition. For example, if h is given by (4) then, as detailed in Subsection 2.2, there is a positive constant ρ such that for all integers k greater than 2

$$\int|\xi|^k\,d\mu(\xi)\leqslant\rho^k k!. \tag{10}$$

In this case we are able to obtain the exponential estimate described in the abstract.

In subsection 2.3 we consider a variant of (10) where $k!$ is replaced by k^{rk}, r an arbitrary real constant. As might be expected, this leads to somewhat different bounds on (9).

Because of the local nature of the result, we restrict our attention to the case where Ω is a cube.

THEOREM 1. *Suppose h is conditionally positive definite of order m and the corresponding measure μ satisfies* (10) *for all k greater than $2m$. Then, given a positive number b_0, there are positive constants δ_0 and λ, $0<\lambda<1$, which depend on b_0 and h for which the following is true: If $f\in\mathscr{C}_{h,m}$ and s is the h spline that interpolates f on a subset X of R^n then*

$$|f(x)-s(x)|\leqslant\lambda^{1/\delta}\|f\|_h$$

holds for all x in a cube E provided that (i) *E has side b and $b\geqslant b_0$,* (ii) $0<\delta\leqslant\delta_0$. *and* (iii) *every subcube of E of side δ contains a point of X.*

Observe that every cube of side δ contains a ball of radius $\delta/2$. Thus the subcube condition is satisfied when $\delta = 2d(E, X)$. More generally, we can easily conclude the following:

COROLLARY 1. *Suppose h satisfies the hypotheses of the theorem, Ω is a set which can be expressed as the union of rotations and translations of a fixed cube of side b_0, and X is a subset of R^n. Then there are positive constants d_0 and λ, $0 < \lambda < 1$, which depend on b_0 and h for which the following is true: If $d \leqslant d_0$, $f \in \mathscr{C}_{h,m}$ and s is the h spline that interpolates f on X then*

$$|f(x) - s(x)| \leqslant \lambda^{1/d} \|f\|_h$$

holds for all x in Ω where $d = d(\Omega, X)$.

Note that any ball in R^n satisfies the hypothesis on Ω in the above corollary. Indeed, any set Ω with sufficiently smooth boundary satisfies this hypothesis.

2. Details for Theorem 1, Examples, and Generalizations

As alluded to in the introduction, Lemma 1 is an important ingredient in the proof of this theorem. The following lemma, which is a transparent consequence of Lemma 1 and routine arguments involving linear functionals, is in convenient form for applying this ingredient.

LEMMA 2. *Let Q, Y, and γ_n be as in Lemma 1. Then, given a point x in Q, there is a measure σ supported on Y such that*

$$\int p(y)\, d\sigma(y) = p(x)$$

for all p in $\mathscr{P}_k$, and

$$\int d|\sigma|(y) \leqslant e^{2n\gamma_n(k+1)}.$$

2.1. *Proof of Theorem* 1

First, let ρ, γ_n, and b_0 be the constants appearing in Inequality (10), Lemma 1, and Theorem 1, respectively. Let

$$B = 2\rho\sqrt{n}\, e^{2n\gamma_n} \qquad \text{and} \qquad C = \max\left\{B, \frac{2}{3b_0}\right\}.$$

Let

$$\delta_0 = \frac{1}{3C\gamma_n(m+1)},$$

where m is the order of conditional positive definiteness of h. We will show that δ_0 as defined above can be used for the constant in the statement of Theorem 1.

For now, let x be any point of the cube E and recall that Theorem 4.2 of [8] implies that

$$|f(x)-s(x)| \leqslant c_k \|f\|_h \int |y-x|^k \, d|\sigma|(y) \tag{11}$$

whenever $k>m$, where σ is any measure supported on X such that

$$\int p(y)\, d\sigma(y) = p(x) \tag{12}$$

for all polynomials p in $\mathscr{P}_{k-1}$. Here

$$c_k = \left\{ \int \frac{|\xi|^{2k}}{(k!)^2} \, d\mu \right\}^{1/2}$$

whenever $k>m$ and by virtue of (9)

$$c_k \leqslant (2\rho)^k. \tag{13}$$

To obtain the desired bound on $|f(x)-s(x)|$ it suffices to find a suitable bound for

$$I = c_k \int |y-x|^k \, d|\sigma|(y).$$

This is done by choosing the measure σ appropriately. We proceed as follows:

Let δ be a parameter as in the statement of the theorem. Since $\delta \leqslant \delta_0$ we may chose an integer k so that

$$1 \leqslant 3C\gamma_n k\delta \leqslant 2. \tag{14}$$

Note that such a k is $\geqslant m+1$ and $\gamma_n k\delta \leqslant b_0$. Let Q be any cube which contains x, has side $\gamma_n k\delta$, and is contained in E. Subdivide Q into $(\gamma_n k)^n$ congruent subcubes of side δ. Since each of these subcubes must contain a point of X, select a point of X from each such subcube and call the resulting discrete set Y. By virtue of Lemma 1 we may conclude that there is a measure σ supported on Y which satisfies (12) and enjoys the estimate

$$\int d|\sigma|(y) \leqslant e^{2n\gamma_n k}. \tag{15}$$

We use this measure in (11) to obtain an estimate on I.

Using (13), (15), and the fact that support of σ is contained in Q whose diameter is $\sqrt{n}\,\gamma_n k\delta$ we may write

$$I \leqslant (2\rho)^k (\sqrt{n}\,\gamma_n k\delta)^k\, e^{2n\gamma_n k} \leqslant (C\gamma_n k\delta)^k. \tag{16}$$

Since

$$C\gamma_n k\delta \leqslant \frac{2}{3} \qquad \text{and} \qquad k \geqslant \frac{1}{3C\gamma_n\delta}$$

Inequality (16) implies that

$$I \leqslant ((2/3)^{1/(3C\gamma_n)})^{1/\delta}.$$

Hence we may conclude that

$$|f(x) - s(x)| \leqslant \lambda^{1/\delta}\, \|f\|_h,$$

where

$$\lambda = (2/3)^{1/(3C\gamma_n)}.$$

2.2. *Examples*

A well known class of examples of conditionally positive definite h's is given by

$$h(x) = \frac{\Gamma(a/2)}{(1+|x|^2)^{a/2}},$$

where a is a fixed real number $\neq 0, -2, -4, \ldots$ and Γ is the classical gamma function. The corresponding measure μ is given by

$$d\mu(\xi) = c_a |\xi|^{(a-n)/2} K_{(n-a)/2}(|\xi|),$$

where c_a is a positive constant and K_ν is a modified Bessel function of the second kind; see [8] for more details and the cases $a = 0, -2, -4, \ldots$. Because of the exponential decay of $K_\nu(t)$ as $t \to \infty$ the moments of μ grow like $\rho^k k!$ and hence μ satisfies (9) whenever k is sufficiently large.

The important example of the Gaussian

$$h(x) = e^{-|x|^2}$$

has corresponding measure

$$d\mu(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/4}\, d\xi,$$

of course. The moments of μ grow like $\rho^k \sqrt{k!}$. Although Theorem 1 provides a bound on the error, in this case one expects better estimates because the growth of these moments is significantly slower than hypothesized.

More generally, consider the case when the measure μ is given by

$$d\mu(\xi) = e^{-|\xi|^a}\, d\xi,$$

where a is a positive constant. Here, of course,

$$h(x) = \int e^{i\langle x, \xi\rangle}\, e^{-|\xi|^a}\, d\xi.$$

The moments of μ grow like $\rho^k k^{rk}$ where $r = 1/a$. The case $a = 2$ is essentially the Gaussian which together with the rest of the cases $a \geqslant 1$ is covered by Theorem 1. On the other hand if $0 < a < 1$ the bound on the rate of growth of the moments hypothesized in the statement of Theorem 1 fails to hold.

The theorems in Subsection 2.3 provide answers to the questions raised above.

2.3. *Generalizations*

As mentioned in the introduction, different bounds on the rate of growth of the moments of the measure μ result in different estimates on the difference between f and its h spline interpolant s off the interpolated set. Here we consider the case

$$\int |\xi|^k\, d\mu(\xi) \leqslant \rho^k k^{rk} \tag{17}$$

for $k > 2m$, where r is a real constant and ρ is a positive constant.

Note that in view of Stirling's formula there are positive constants ρ_1 and ρ_2 such that

$$\rho_1^k k^k \leqslant k! \leqslant \rho_2^k k^k. \tag{18}$$

Thus the case $r = 1$ was treated in Theorem 1. Also observe that Theorem 1 provides an estimate in the case $r > 1$. However it is possible to get a more sensitive estimate in this case without much more work; this is shown in Theorem 3 and its proof. We first consider the case $r \geqslant 1$.

THEOREM 2. *Suppose h is conditionally positive definite of order m and the corresponding measure μ satisfies* (17) *with $r \geqslant 1$ for all k greater than $2m$. Then, given a positive number b_0, there are positive constants δ_0 and λ, $0 < \lambda < 1$, which depend on h, r, and b_0 and for which the following is true: If $f \in \mathscr{C}_{h,m}$ and s is the h spline that interpolates f on a subset X of R^n then*

$$|f(x) - s(x)| \leqslant \lambda^{\delta^{-1/r}} \|f\|_h$$

holds for all x in a cube E provided that (i) *E has side b and $b \geqslant b_0$,* (ii) *$0 < \delta \leqslant \delta_0$, and* (iii) *every subcube of E of side δ contains a point of X.*

Proof. In view of (17) and (18) there is a constant ρ_0 such that

$$\frac{1}{k!} \int |\xi|^k \, d\mu(\xi) \leqslant \rho_0^k k^{(r-1)k}.$$

Let γ_n and b_0 be the constants appearing in the statements of Lemma 1 and Theorem 2 respectively. Let

$$B = 2\rho_0 \sqrt{n}\, e^{2n\gamma_n} \qquad \text{and} \qquad C = \max\left\{B, \frac{2}{3b_0}\right\}.$$

and let

$$\delta_0 = \frac{1}{3^r C \gamma_n (m+1)^r},$$

where m is the order of conditional positive definiteness of h. Let δ be a parameter as in the statement of the theorem. Since $\delta \leqslant \delta_0$, $3(C\gamma_n\delta)^{1/r}$ is less than 1 and we may choose an integer k such that

$$1 \leqslant 3(C\gamma_n\delta)^{1/r} k \leqslant 2.$$

Note that such a k is $\geqslant m+1$ and $\gamma_n k\delta \leqslant b_0$.

Proceeding as in the proof of Theorem 1 we get

$$|f(x) - s(x)| \leqslant I \|f\|_h, \tag{19}$$

where

$$I \leqslant \rho_0^k k^{(r-1)k} (\sqrt{n}\, \gamma_n k\delta)^k e^{2n\gamma_n k} \leqslant ((C\gamma_n\delta)^{1/r} k)^{rk}.$$

Since

$$(C\gamma_n\delta)^{1/r}k \leqslant \frac{2}{3} \qquad \text{and} \qquad k \geqslant \frac{1}{3(C\gamma_n\delta)^{1/r}}$$

we may conclude that

$$I \leqslant ((2/3)^{1/(3(C\gamma_n)^{1/r})})^{\delta^{-1/r}}.$$

In view of (19) the theorem now follows with

$$\lambda = (2/3)^{1/(3(C\gamma_n)^{1/r})}. \quad \blacksquare$$

THEOREM 3. *Suppose h is conditionally positive definite of order m and the corresponding measure μ satisfies* (17) *with $r < 1$ for all k greater than* $2m$. *Then, given a positive number b_0, there are positive constants δ_0, c,* and *C, which depend on h, r, and b_0 and for which the following is true: If $f \in \mathscr{C}_h$ and s is the h spline that interpolates f on a subset X of R^n then*

$$|f(x) - s(x)| \leqslant (C\delta)^{c/\delta} \|f\|_h$$

holds for all x in a cube E provided that (i) *E has side b and $b \geqslant b_0$,* (ii) *$0 < \delta \leqslant \delta_0$, and* (iii) *every subcube of E of side δ contains a point of X.*

Proof. Let

$$\delta_0 = \min \left\{ \frac{1}{(Bb_0^r)^{1/(1-r)}\gamma_n}, \frac{b_0}{2\gamma_m(m+1)} \right\},$$

where γ_n is the constant defined in Lemma 1, and

$$B = 2\rho_0 \sqrt{n}\, e^{2n\gamma_n}$$

with ρ_0 as in the proof of Theorem 2. Then if $\delta \leqslant \delta_0$ there is an integer k such that

$$\frac{b_0}{2} \leqslant \gamma_n \delta k \leqslant b_0.$$

Arguing as in the proof of Theorem 2 we can conclude that

$$|f(x) - s(x)| \leqslant I \|f\|_h, \tag{20}$$

where $I \leqslant (B\gamma_n\,\delta k^r)^k$. Since $k \leqslant b_0/(\gamma_n\delta)$ we may write

$$I \leqslant \left(B\gamma_n\delta \left(\frac{b_0}{\gamma_n\delta}\right)^r \right)^k$$

and since $B\gamma_n^{1-r} b_0^r \delta^{1-r} \leqslant 1$ and $k \geqslant b_0/(2\gamma_n \delta)$ it follows that

$$I \leqslant (B\gamma_n^{1-r} b_0^r \delta^{1-r})^{b_0/(2\gamma_n \delta)}.$$

The last inequality together with (20) implies the desired result with

$$C = (Bb_0^r)^{1/(1-r)} \gamma_n \qquad \text{and} \qquad c = \frac{(1-r)\, b_0}{2\gamma_n}. \quad \blacksquare$$

3. Details for Lemma 1

We begin by asserting that it suffices to prove

$$\sup_{x \in Q} |p(x)| \leqslant e^{2qn} \sup_{y \in Y} |p(y)|. \tag{21}$$

If $q = \gamma_n(k+1)$ this inequality is identical to that in Lemma 1. To see why (21) is sufficient, define q' by $q' = \gamma_n(k+1)$ and let $Q' \subset Q$ be a cube that contains exactly $(q')^n$ of the q^n subcubes of Q. By (21) we have

$$\sup_{Q'} |p| \leqslant e^{2q'n} \sup_{Q' \cap Y} |p|. \tag{22}$$

The inequality in Lemma 1 now follows because $\sup_{Q' \cap Y} |p| \leqslant \sup_Y |p|$ and every point in Q lies in at least one such Q'. Our proof will actually establish

$$\sup_{x \in Q} |p(x)| \leqslant \left(\frac{(2q)^k}{k!} \right)^n \sup_{y \in Y} |p(y)|. \tag{23}$$

This gives (21) because $(2q)^k/k! \leqslant e^{2q}$.

To simplify notation we assume $Q = [0, 1]^n$. To see that this involves no loss of generality, let Q be any cube in R^n and let ϕ be an affine transformation mapping $[0, 1]^n$ onto Q. Then polynomials p on Q are related to polynomials f on $[0, 1]^n$ via the correspondence

$$f(x) = p(\phi(x))$$

and the corresponding subdivisions and discrete subsets Y are related analogously. It is clear that an estimate like that given by Lemma 1 on the size of f on $[0, 1]^n$ implies the corresponding estimate on the size of p on Q.

Our proof of Lemma 1 involves induction on the dimension n. While Lemma 1 and its proof are elementary and well known in the case $n = 1$, in the first subsection we formulate it in a manner convenient for the necessary induction argument. The general case involves certain unpleasant combinatoric and geometric complications, so for the sake of clarity, we spell out the argument in the case $n=2$ in the second subsection. The general case is considered in the third subsection.

3.1. *The Case* $n=1$

PROPOSITION 1. *Let $T=\{t_0, ..., t_k\}$ be a subset of the unit interval $[0,1]$ and assume $t_{i-1}+1/q \leqslant t_i$, for $i=1, ..., k$. Then for all $p \in \mathcal{P}_k$,*

$$\sup_{t \in [0,1]} |p(t)| \leqslant \frac{(2q)^k}{k!} \sup_{t \in T} |p(t)|.$$

Proof. Recall $p=\sum_{i=0}^k p(t_i)\, L_i$, where

$$L_i(t)=\prod_{j=0, j\neq i}^{k} \frac{t-t_j}{t_i-t_j}.$$

The assumption $1/q \leqslant t_i - t_{i-1}$ implies $|t_i - t_j|^{-1} \leqslant q/|i-j|$. Also, $|t-t_j| \leqslant 1$ for all $t \in [0, 1]$. Hence, for such t, $|L_i(t)| \leqslant q^k/[i!(k-i)!]$ and

$$\sum_{i=0}^k |L_i(t)| \leqslant q^k \sum_{i=0}^k \frac{1}{i!\,(k-i)!} = \frac{(2q)^k}{k!},$$

which gives the desired inequality. ∎

3.2. *The Case* $n=2$

PROPOSITION 2. *Suppose the square $Q=[0, 1]^2$ is divided into q^2 identical subsquares and X is a set that intersects each subsquare. If $q \geqslant 12(k+1)$, then for all $p \in \mathcal{P}_k$*

$$\sup_{x \in Q} |p(x)| \leqslant \left(\frac{(2q)^k}{k!}\right)^2 \sup_{x \in X} |p(x)|. \tag{24}$$

Proof. Instead of (24) we show that if $h \in \mathcal{P}_k$ and $|h(x)|<1$ for all $x \in X$ then

$$\sup_Q |h| \leqslant \left(\frac{(2q)^k}{k!}\right)^2. \tag{25}$$

That this implies (24) can be seen by considering $h=p/(\varepsilon+\sup_X |p|)$, $\varepsilon>0$.

Let Q_i, $i \in I$ denote the q^2 subsquares of Q and set $m_i=\min_{\partial Q_i} |h|$, where ∂Q_i denotes the boundary of Q_i. Let N_0 be the number of points in $I_0=\{i \in I\colon m_i<1\}$. We assert that

$$N_0 \geqslant q^2-(2k-1)^2. \tag{26}$$

To see this, take $b=(b_1, b_2)$, let $g_b(x)=|h(x)|^2+(b_1 x_1+b_2 x_2)$, and note that for every $i \in I \backslash I_0$,

$$\min_{Q_i} g_0 < 1 \leqslant (m_i)^2 = \min_{\partial Q_i} g_0.$$

Thus we can choose $\varepsilon > 0$ such that if $|b| < \varepsilon$ then for every $i \in I \backslash I_0$

$$\min_{Q_i} g_b < \min_{\partial Q_i} g_b.$$

When this occurs, g_b has a critical point in the interior of Q_i. Such a b can be chosen such that all the critical points of g_b are nondegenerate; for example see Lemma 6.2 on p. 40 of [9]. Now $g_b \in \mathscr{P}_{2k}$, so by virtue of Proposition 4 in Subsection 3.4 it can have at most $(2k-1)^2$ nondegenerate critical points. Thus $I \backslash I_0$ has at most $(2k-1)^2$ points and (26) follows.

For each $i \in I_0$ select a point $y_i \in \partial Q_i$ such that $|h(y_i)| < 1$ and that y_i is not one of the four corners of Q_i. Partition I_0 into four subsets $I_1, ..., I_4$ according to whether y_i lies on the top, bottom, left, or right edge of Q_i. Let N_1 be the number of points in I_1 and assume without loss of generality that $N_1 \geqslant N_0/4$.

For each $j=1, ..., q$ let $I(j)$ be the set of i's for which Q_i lies in the horizontal strip

$$\{(t, s): 0 \leqslant t \leqslant 1, (j-1) \leqslant qs \leqslant j\}.$$

Let $N(j)$ be the number of points in $I_1 \cap I(j)$ and let N be the number of points in $J=\{j: N(j) \geqslant 2(k+1)\}$.

Noting $N_1 = \sum_{j=1}^{q} N(j) \leqslant Nq+(q-N)(2k+1) = q(2k+1)+(q-2k-1)N$, we observe that $N \leqslant k$ would imply

$$N_1 \leqslant q(3k+1)-k(2k+1).$$

Since this gives $N_0 \leqslant 4N_1 \leqslant q(12k+4)-(8k^2+4k)$ which violates (26), we conclude that $N \geqslant k+1$.

Let $p_j(t)=h(t, j/q)$. In $N(j)$ of the intervals

$$\frac{r-1}{q} \leqslant t \leqslant \frac{r}{q}, \qquad r=1, ..., q$$

there is a point t with $|p_j(t)| < 1$. If $j \in J$ there are at least $2(k+1)$ such intervals. Thus we can apply Proposition 1 to p_j and see that

$$\max_{t \in [0,1]} |p_j(t)| \leqslant \frac{(2q)^k}{k!} \tag{27}$$

for every $j \in J$. Using this and the fact that J has $N \geqslant k+1$ points we can apply Proposition 1 again, this time to $p(s)=h(a, s)$, $a \in [0, 1]$ to arrive at (25). ▌

3.3. *The General Case*

PROPOSITION 3. *Define γ_n for $n=1, 2, \ldots$ by $\gamma_1=2$, $\gamma_n=2n(1+\gamma_{n-1})$, $n>1$. Let $r \in (0, 1]$ and let k and q be positive integers with $q \geqslant \gamma_n(k+1)/r$. Subdivide the unit n-cube $[0, 1]^n$ into q^n identical subcubes and let N be the number of such subcubes that intersect a subset X of R^n. If $N \geqslant rq^n$ then for all $f \in \mathscr{P}_k$*

$$\sup_{x \in [0,1]^n} |f(x)| \leqslant \left(\frac{(2q)^k}{k!}\right)^n \sup_{x \in X} |f(x)|. \tag{28}$$

Proof. We first deal with the case $n=1$. In that case the subcubes are the intervals $I_i=[(i-1)/q, i/q]$, $i=1, \ldots, q$. Let $i(1)<i(2)< \cdots <i(N)$ give the intervals that intersect X. For each $j=1, \ldots, N$ choose $x(j) \in I_{i(j)} \cap X$. By assùmption, $N \geqslant rq \geqslant 2(k+1)$. The points

$$t_0=x(1),\ t_1=x(3), \ldots, t_k=x(1+2k)$$

satisfy $t_j-t_{j-1} \geqslant 1/q$ so (28) follows from Proposition 1.

To complete the proof we use induction on n. The integers k and q will be held fixed during the induction. Let $n'=n-1$ and define r' by $\gamma_{n'}/r'=\gamma_n/r$. Then $q \geqslant \gamma_{n'}(k+1)/r'$. Subdivide the unit n'-cube $[0, 1]^{n'}$ into $q^{n'}$ identical subcubes and let N' be the number of such subcubes that intersect $X' \subset R^{n'}$. If $N' \geqslant r'q^{n'}$ then, by induction, for all $g \in \mathscr{P}_k$

$$\sup_{[0,1]^{n'}} |g| \leqslant \left(\frac{(2q)^k}{k!}\right)^{n'} \sup_{X'} |g|. \tag{29}$$

Instead of (28) we will show that if $h \in \mathscr{P}_k$ and $|h(x)|<1$ for all $x \in X$ then

$$\sup_{[0,1]^n} |h| \leqslant \left(\frac{(2q)^k}{k!}\right)^n. \tag{30}$$

That this implies (28) can be seen by considering $h=p/(\varepsilon+\sup_X |p|)$, $\varepsilon>0$.

Let $\mathscr{Q}$ denote the family of q^n subcubes of $[0, 1]^n$. For each $Q \in \mathscr{Q}$ let $m_Q=\min_{\partial Q} |h|$ where ∂Q denotes the boundary of Q. Let

$$\mathscr{Q}_h=\{Q \in \mathscr{Q}: m_Q<1\}. \qquad \mathscr{Q}_X=\{Q \in \mathscr{Q}: Q \cap X \neq \varnothing\}.$$

Note that N is the number of elements in $\mathscr{Q}$ and let N_h be the number of elements in $\mathscr{Q}_h$. We assert that

$$N_h \geqslant N-(2k)^n. \tag{31}$$

To see this, for $b \in R^n$ consider the functions g_b defined by

$$g_b(x) = |h(x)|^2 + (b_1 x_1 + \cdots + b_n x_n).$$

If $\mathcal{Q} \in \mathcal{Q}_X \backslash \mathcal{Q}_h$ then

$$\min_Q g_0 < 1 \leqslant \min_{\partial Q} g_0.$$

Thus we can choose $\varepsilon > 0$ so that for all $Q \in \mathcal{Q}_X \backslash \mathcal{Q}_h$ and all $|b| < \varepsilon$

$$\min_Q g_b < \min_{\partial Q} g_b.$$

When this holds, it is evident that g_b has a critical point in the interior of Q. Thus g_b has at least $N - N_h$ critical points. Such a b can be chosen such that the critical points of g_b are nondegenerate, see Lemma 6.2 on page 40 of [9]. Since $g_b \in \mathscr{P}_{2k}$, by virtue of Proposition 4 in Subsection 3.4 it can have at most $(2k-1)^n$ nondegenerate critical points. Thus $N - N_h \leqslant (2k-1)^n$ which gives (31).

For each $Q \in \mathcal{Q}_h$ a point $y(Q) \in \partial Q$ can be selected so that $|h(y(Q))| < 1$. By moving $y(Q)$ slightly, if necessary, it may also be assumed that $y(Q)$ lies on exactly one of the hyperplanes

$$M_{mj} = \{ y \in R^n \colon y_m = j/q \}, \qquad m = 1, \ldots, n,\ j = 0, \ldots, q.$$

Let $N_h(m, j)$ be the number of Q's for which $y(Q) \in M_{m,j}$. Let $N_{h,m} = \sum_{j=0}^{q} N_h(m, j)$, and note that $N_h = \sum_{m=1}^{n} N_{h,m}$. Without loss of generality we assume $N_{h,n} \geqslant N_h/n$.

Let $Y = \{ y(Q) \colon Q \in \mathcal{Q}_h \}$, $Y_j = Y \cap M_{n,j}$. In each hyperplane $M_{n,j}$ there are q^{n-1} $(n-1)$-cubes that correspond to the subdivision of $[0, 1]^n$ into q^n n-cubes. Let $N(Y_j)$ be the number of $(n-1)$-cubes in $M_{n,j}$ that intersect Y_j. Then $N(Y_j) \geqslant N_h(n, j)/2$ because for each $(n-1)$-cube Q' in $M_{n,j}$ there are at most two n-cubes $Q \in \mathcal{Q}$ which contain Q'. Thus we have

$$2\left(\sum_{j=0}^{q} N(Y_j)\right) \geqslant N_{h,n} \geqslant N_h/n. \tag{32}$$

If $N(Y_j) \geqslant r' q^{n-1}$ then from (29) we get

$$|h(x', j/q)| \leqslant \left(\frac{(2q)^k}{k!}\right)^{n-1} \sup_{Y_j} |h| < \left(\frac{(2q)^k}{k!}\right)^{n-1} \tag{33}$$

for all $x' \in [0, 1]^{n-1}$. Let $J = \{ j \colon N(Y_j) \geqslant r' q^{n-1} \}$. We will show below that

J has at least $k+1$ elements. This allows us to apply Proposition 1 to $p(t)=h(x', t)$. The result is

$$|h(x', t)| \leqslant \frac{(2q)^k}{k!} \max_{j \in J} |h(x', j/q)|$$

for every $t \in [0, 1]$. Because of (33), this gives (30).

Let s be the number of elements in J. It remains to show $s \geqslant k+1$. For all j, $N(Y_j) \leqslant q^{n-1}$ and for $j \notin J$, $N(Y_j) < r'q^{n-1}$. Thus

$$\sum_{j=0}^{q} N(Y_j) \leqslant sq^{n-1} + (1+q-s)\, r'q^{n-1}.$$

Combining this with (32), (31), and the hypothesis $N \geqslant rq^n$ gives

$$\frac{1}{2n}(rq^n - (2k)^n) \leqslant \sum_{j=0}^{q} N(Y_j) \leqslant sq^{n-1} + (1+q-s)\, r'q^{n-1}$$

or, after division by q^{n-1},

$$\frac{rq}{2n} - \frac{(2k)^n q}{q^n 2n} - (1+q)\, r' \leqslant s(1-r'). \tag{34}$$

By definition of r', $r = r'\gamma_n/\gamma_{n-1}$ with $\gamma_n = 2n(1+\gamma_{n-1})$. Hence $r/2n = r'(1+\gamma_{n-1})/\gamma_{n-1}$ or $r/2n - r' = r'/\gamma_{n-1}$. Thus (34) can be rewritten as

$$\frac{r'q}{\gamma_{n-1}} - \left(\frac{(2k)^n q}{q^2 2n} + r'\right) \leqslant s(1-r').$$

By assumption we have $q \geqslant \gamma_n(k+1)/r = \gamma_{n-1}(k+1)/r'$. Taking $M = \gamma_{n-1}/r'$ in the following lemma, we find $(1-r')(k+1) \leqslant s(1-r')$ which gives $s \geqslant k+1$. ∎

LEMMA 3. *If* $n \geqslant 2$, $k \geqslant 1$, $r' \in (0, 1]$, $Mr' \geqslant 2$, *and* $q \geqslant M(k+1)$ *then*

$$(1-r')(k+1) \leqslant \frac{q}{M} - \frac{(2k)^n q}{q^n 2n} - r'. \tag{35}$$

Proof. From $k \leqslant k+1 \leqslant q/M$ we have $k/q \leqslant 1/M \leqslant 1/2$ and

$$\left(\frac{2k}{q}\right)^n \frac{M(k+1)}{2n} \leqslant \left(\frac{2}{M}\right)^2 \frac{M(k+1)}{2n} \leqslant \frac{k+1}{M} \leqslant \frac{2k}{M} \leqslant kr'.$$

Multiplying this by -1 and then adding $1+k$ gives

$$1+k-kr' \leqslant \left(\frac{1}{M}-\left(\frac{2k}{q}\right)^n \frac{1}{2n}\right) M(k+1).$$

Hence

$$1+k-kr' \leqslant \left(\frac{1}{M}-\left(\frac{2k}{q}\right)^n \frac{1}{2n}\right) q$$

which is the same as (35). ∎

3.4. *Critical points of polynomials*

PROPOSITION 4. *If p is a real valued polynomial on R^n of degree d then p can have at most $(d-1)^n$ nondegenerate critical points.*

Proof. A simple argument for the case $n=2$ goes as follows: Let q be the greatest common factor of $\partial p/\partial x_1$ and $\partial p/\partial x_2$, and write $\partial p/\partial x_i = qp_i$, $i=1, 2$. If q vanishes at x_0 then

$$\frac{\partial^2 p}{\partial x_i\, \partial x_j}(x_0) = p_j(x_0)\frac{\partial q}{\partial x_i}(x_0), \qquad \text{so} \quad \det\left(\frac{\partial^2 p}{\partial x_i\, \partial x_j}(x_0)\right) = 0.$$

Hence x_0 is a degenerate critical point. At any nondegenerate critical point x_0 we therefore have $p_1(x_0)=0=p_2(x_0)$. Since p_1 and p_2 have no common factor, the two-variable version of Bezout's theorem, for example see [12], implies that the number of such points x_0 does not exceed $N=(\deg p_1)(\deg p_2) \leqslant (d-1)^2$. The lack of such a convenient form of Bezout's theorem when $n>2$ is what makes the general case more difficult.

To obtain a proof in the general case we begin by observing that it is a corollary of its complex analogue. Indeed, there is a unique $P \in \mathscr{P}_d(C^n)$ such that $p(x) = P(x+\imath 0)$ for all $x \in R^n$. Here and in what follows $\imath = \sqrt{-1}$. From

$$\frac{\partial p}{\partial x_k}(x) = \frac{\partial P}{\partial z_k}(x+\imath 0)$$

and the corresponding formula for second order partial derivatives, it is clear that if x_0 is a nondegenerate critical point of p then $z_0 = x_0 + \imath 0$ is a nondegenerate critical point of P. Thus the general case follows from the next proposition. ∎

PROPOSITION 5. *If $p \in \mathscr{P}_d(C^n)$ then p can have at most $(d-1)^n$ nondegenerate critical points.*

Proof. For $j=1, ..., n$ let $p_j = \partial p/\partial z_j$. All critical points of p are degenerate if $p_j = 0$, so we assume $p_j \neq 0$ for all j. Let $m = \dim P_d(C^n)$; we identify points $c \in C^m$

$$c = (c_\alpha)_{|\alpha| \leqslant d} = (a_\alpha + \imath b_\alpha)_{|\alpha| \leqslant d} = a + \imath b \tag{36}$$

with points $(a, b) \in R^{2m}$. For $z_0 \in C$, $z \in C^n$ and $c \in C^m$ let

$$f(z_0, z, c) = \sum_{|\alpha| \leqslant d} c_\alpha z^\alpha z_0^{d-|\alpha|}.$$

Let c_p be the point in C^m such that $p(z) = f(1, z, c_p)$ for all $z \in C^n$. Note that $p_j(z) = f_j(1, z, c_p)$ where $f_j = \partial f/\partial z_j$, $j = 1, ..., n$.

Let $z(1), ..., z(N)$ be nondegenerate critical points of p. Put $\xi^{(r)} = (1, z(r))$, $r = 1, ..., N$ and observe that $\mathbf{z} = \lambda \xi^{(r)}$, $\lambda \in C$ is a solution of the system $f_j(\mathbf{z}, c_p) = 0$, $j = 1, ..., n$. By Bezout's Theorem [11], if n homogeneous equations $f_j(\mathbf{z}) = 0$ in $n+1$ variables $\mathbf{z} = (z_0, z)$ have only a finite number of solution rays $\mathbf{z} = \lambda \xi^{(r)}$, $r = 1, ..., q$, $\xi^{(r)} \in C^{n+1} \setminus \{0\}$, then $q \leqslant (d-1)^n$, where $d-1$ is the degree of f_j, $j = 1, ..., n$. The desired conclusion, $N \leqslant (d-1)^n$, would follow if we knew that the system $f_j(\mathbf{z}, c_p) = 0$, $j = 1, ..., n$ had only a finite number of solution rays. The latter may not be true, but it suffices to show that we can perturb c_p to obtain a point $c \in C^m$ for which the number, q_c, of solution rays of the system $f_j(\mathbf{z}, c) = 0$, $j = 1, ..., n$ is finite and satisfies $q_c \geqslant N$.

First we show that $q_c \geqslant N$ is automatic if c is close enough to c_p. Consider the map T from $C^n \times C^m$ to C^n given by

$$T(z, c) = (f_1(1, z, c), ..., f_n(1, z, c)).$$

The points $z(i)$ are nondegenerate, so the $n \times n$ matrix $\partial T/\partial z$ is nonsingular at $(z(i), c_p)$, $i = 1, ..., N$. By the Implicit Function Theorem there are analytic functions ζ_i on a neighborhood $B \subset C^m$ of c_p such that

$$T(\zeta_i(c), c) = 0, \qquad \zeta_i(c_p) = z(i), \qquad i = 1, ..., N.$$

By making B smaller, if necessary, it may be assumed that $\zeta_i(c) \neq \zeta_j(c)$ for all $c \in B$ and all $i \neq j$. It is then evident that $q_c \geqslant N$ for all $c \in B$.

To complete the proof we establish that for almost every point $(a, b) \in R^{2m}$, the system $f_j(\mathbf{z}, a + \imath b) = 0$, $j = 1, ..., n$ has only a finite number of solution rays. For $k = 0, ..., n$ define maps J^k from R^{2n} to $\{\mathbf{z} \in C^{n+1} : \mathbf{z}_k = 1\}$ by

$$J^k(x_1, ..., x_{2n}) = (x_1 + \imath x_n, ..., x_k + \imath x_{n+k}, 1, x_{k+1} + \imath x_{n+k+1}, ..., x_n + \imath x_{2n}).$$

Let $V(k, a, b) = \bigcap_{j=1}^n \{x \in R^{2n} : f_j(J^k(x), a + \imath b) = 0\}$. The maps J^k provide coordinate systems for complex projective n space. By compactness of that space, it suffices to prove that $V(k, a, b)$ consists of isolated points for every $k=0, ..., n$ and almost all $(a, b) \in R^{2n}$.

The proof of this uses a theorem from [9]. To prepare, define $f_{j,\alpha}(\mathbf{z})$, $|\alpha| \leqslant d$ by

$$f_{j,\alpha}(z_0, z) = (z_0)^{d-|\alpha|} \frac{\partial z^\alpha}{dz_j} = \frac{\partial f_j}{\partial c_\alpha}(z_0, z, c)$$

and identify $f_{j,\alpha}$ with an $n \times m$ matrix.

We assert that $f_{j,\alpha}(J^k(x))$ has rank n for every $k=0, ..., n$ and every $x \in R^{2n}$. For $k \neq 0$ we take

$$\alpha = \alpha(i, k) = e(i) + (d-1)\, e(k), \qquad i = 1, ..., n,$$

where $\{e(1), ..., e(n)\}$ is the standard basis for R^n, and consider the $n \times n$ matrix $F_{j,i}(x, k) = f_{j,\alpha(i,k)}(J^k(x))$. Then $F_{k,k}(x, k) = d$, $F_{j,j}(x, k) = 1$ for $j \neq k$, and $F_{j,i}(x, k) = 0$ for $j \neq i$, $i \neq k$. It follows that $\det(F_{j,i}(x, k)) = d$, $k \neq 0$, the off diagonal entries of the kth column of F are not needed for this. For $k=0$, the $n \times n$ matrix $F_{j,i}(x, 0) = f_{j,e(i)}(J^0(x))$ is seen to be $\delta_{i,j}$ and our assertion is verified.

To obtain notation more like [9] we fix $k \in \{0, ..., n\}$ and define real valued function $U_1, ..., U_{2n}$ by

$$U_j(x, a, b) + \imath U_{j+n}(x, a, b) = f_j(J^k(x), a + \imath b).$$

Using the analysis of $F_{j,i}(x, k)$ above, we see that the $2n \times 2(n+m)$ matrix of partial derivatives of $U_1, ..., U_{2n}$ has rank $2n$. By Theorem 7.1 on p. 50 of [9] we conclude that for almost all $(a, b) \in R^{2n}$, the $2n \times 2n$ matrix $(\partial U_i/\partial x_j)(x, a, b)$ is nonsingular at every point in

$$V(k, a, b) = \bigcap_{i=1}^{2n} \{x \in R^{2n} \colon U_i(x, a, b) = 0\}.$$

Thus for such (a, b) the points in $V(k, a, b)$ are isolated. ∎

4. Miscellaneous Remarks

A detailed account of conditionally positive definite function and distributions can be found in [6]. For a development of the variational theory which does not involve Fourier transforms see [7]; this paper also contains error estimates which are different from those considered here.

The analogues of Corollary 1 for Theorem 2 and 3 are clear. It is also clear that the analogues of Lemma 1 and the Theorems hold when the cubes are replaced by more general parallelepipeds; simply apply an appropriate affine transformation. Thus analogues of Corollary 1 hold when Ω satisfies an interior cone condition. Since our results seem to apply to most reasonable situations we refrain from exploring further generalizations.

If the measure μ satisfies (17) with $r \leqslant 0$ then it must have compact support. Also recall that in this case the constant C can be taken to be independent of b_0. Since the exponent c is $(1-r)\, b_0/(2\gamma_n)$, if δ is such that $C\delta < 1$, letting $b_0 \to \infty$ it is clear that $|f(x) - s(x)| \to 0$. In other words, for sufficiently small δ if the intersection of X with any cube of side δ is not empty then $s(x) = f(x)$ on R^n. This means, of course, that the values of f on X uniquely determine f. The implications of this to irregular sampling theory, such as that found in [3] or [5] for example, will be explored elsewhere.

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